The compactification of a Lorentz space and some remarks on the foundation of the Theory of Conformal Relativity

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Introduction

Let Lo_n be the real Lorentz space of n dimensions with the metric

$$s^2 = h(x, x) = -x_1^2 - x_2^2 - \cdots - x_{n-1}^2 + x_n^2 \qquad (n \geqq 2). \tag{1}$$

In Conformal Relativity, the fundamental group is the group C_n of all conformal transformations of Lo_n into itself (J. A. Schouten [1]). Among these there is e.g. the inversion

$$(x-a) \rightarrow \frac{(x-a)}{h(x-a, x-a)} \tag{2}$$

in a "sphere" with centre a and radius 1. The fact that this transformation is not defined in the entire space shows that the definition of C_n requires a discussion.

In the first place, we have the theorem of Liouville[1] which says that, if $n > 2$, any locally conformal mapping of Lo_n into itself is a product of inversions and hence transforms circles into circles. If we add this last property as a requirement on a locally conformal transformation, Liouville's theorem is true also when $n = 2$. Still the fact remains that an inversion is not defined everywhere on Lo_n.

However, it is possible to imbed Lo_n conformally into a compact Riemannian space on which inversions are globally conformal. How this can be done will be shown in this paper. The procedure, which makes use of the fact that conformal transformations can be dealt with as orthogonal transformations in $n+2$ dimensions, is well known in principle (Weyl [2]), but as far as I know, the geometrical significance of the compactifical has not been studied before.

I shall now describe briefly how conformal transformations can be obtained from orthogonal transformations. Let R_{n+2} be the real $(n+2)$-dimensional space with the metric

$$H(y, y) = h(y, y) - y_{n+1}^2 + y_{n+2}^2 \tag{3}$$

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[1] No modernized proof of this theorem seems to exist.

and let P_{n+1} be the corresponding projective space, and K_n the projective manifold $H(y, y) = 0$. It is easy to see that the mapping $x \to Tx = y$, defined by putting y proportional to the vector

$$[x_1, \ldots x_n, -\tfrac{1}{2}(1+s^2), \tfrac{1}{2}(1-s^2)], \tag{4}$$

maps Lo_n upon the part of K_n where $y_{n+1} \neq y_{n+2}$. The y_i are sometimes called homogeneous "*hexaspherical*" coordinates (F. Klein [3]). That $H(y, y) = 0$ for the vector (4) is an immediate consequence of (1). Let $\mathcal{G}_{n+2}$ be the group of all linear homogeneous H-orthogonal transformations of R_{n+2} into itself. Any element g of $\mathcal{G}_{n+2}$ transforms K_n into itself, and induces a conformal transformation of Lo_n into itself in the sense that $T^{-1}gT$ is such a transformation. More precisely, reflection in the plane

$$Pl(y) = a_1 y_1 + a_2 y_2 + \cdots + a_{n+2} y_{n+2} = 0, \quad a_{n+1} + a_{n+2} \neq 0 \tag{5}$$

corresponds to an inversion in the sphere

$$a_1 x_1 + a_2 x_2 \cdots + a_n x_n - \tfrac{1}{2} a_{n+1}(1+s^2) + \tfrac{1}{2} a_{n+2}(1-s^2) = 0. \tag{6}$$

We also note that each g of $\mathcal{G}_{n+2}$ transforms a linear relation (5) between y_n into a linear relation between y_i', which of course means that a "sphere" is mapped on a "sphere" by the corresponding transformation $T^{-1}gT$ in Lo_n. Each Lorentz sphere S in Lo_n is defined by an equation $h(x-a, x-a) = -r^2$ and we can distinguish between space-like, time-like and null-spheres.

Two elements g and g' of $\mathcal{G}_{n+2}$ induce the same conformal transformation of Lo_n, if and only if $g' = \pm g$. Hence the natural way of defining $\mathcal{C}_n$ is to identify it with the quotient group

$$\mathcal{G}_{n+2}/\{E, -E\}, \tag{7}$$

where E is the unit element of $\mathcal{G}_{n+2}$. The elements of $\mathcal{G}_{n+2}$ which correspond to inversions in the origin in Lo_n will be given explicitly below. The connectivity properties of the quotient group will also be discussed.

If we accept the theorem of Liouville, we understand that every locally conformal mapping has an extension to the entire space, as the inversions determining a locally defined transformation in Lo_n can be converted to transformations of $\mathcal{G}_{n+2}$.

Let us now pass to the compactification of Lo_n. To begin with, consider the manifold K_n, which we may think of as the intersection of the sphere $y_1^2 + y_2^2 \ldots + y_{n+2}^2 = 1$ and the cone $H(y, y) = 0$ with diametrically opposite points identified. This corresponds to the usual topological picture of projective space. The metric H in R_{n+2} induces a Riemannian metric on K_n, namely

$$H(dy, dy), \tag{8}$$

where we assume that dy is expressed in terms of coordinates on K_n. Actually, we can also choose any other metric which is proportional to this one. We now observe two fundamental facts which are easy to show, namely that every element g of $\mathcal{G}_{n+2}$ induces a conformal mapping of K_n into itself, and that T is a conformal mapping of Lo_n into K_n. Hence the natural way to compactify Lo_n is to add to it elements corresponding to the part of K_n which is not covered by the map $T(Lo_n)$ of Lo_n, i.e. to add the part of K_n where $y_{n+1} = -y_{n+2}$. This part consists of a "cone" of closed null-geodesics issuing from the point $p = (0, 0, \ldots, 1, 1)$ of K_n and is simply the projective intersection of the non-projective cone $H(y, y) = 0$ with its tangent plane through p. Any point p on K_n is, of course, the vertex of such a cone. Let us consider its image in Lo'_n. The results of the analysis in this paper can be stated as follows ($P = T^{-1}(p)$ is the image in Lo_n, viz. Lo'_n of the point p on K_n):

(i) The cone of P_∞ consists of all the ideal elements I of Lo'_n and can be called "the light cone at infinity".

(ii) If $P \neq P_\infty$ but still $P \subset I_\infty$, its cone is a singular hyperplane Pl in Lo_n, or in other words a tangent plane to the light cone in each point of Pl.

(iii) If P is not on I_∞, its cone is an ordinary light cone in Lo_n through P.

Further geometrical properties are

(iv) Each non-isotropic straight line in Lo_n is a closed curve in Lo'_n passing through P.

(v) Each null-ray in Lo_n is a closed curve in Lo'_n passing through a well-determined point on I_∞. Two null-rays determine the same point on I_∞ if and only if they are parallel and belong to the same singular hyperplane. At first sight, it could be expected that all parallel null-rays should determine the same point on I_∞, but that is a mistake.

(vi) The compactified space Lo'_n is orientable if n is even, and not orientable if n is odd (this property parallels the orientability of projective spaces).

In § 6 we carry through the general constructions referred to above in more detail for the two-dimensional case, and a reader who wishes more intuitive material may consult that section before reading the rest of the paper.

As a matter of curiosity we shall show in § 7 that in the two-dimensional case all Lorentz-conformal transformations can be represented in the form

$$w = \frac{a z + b}{c z + d} \tag{9}$$

just as the Euclid-conformal transformations of an Euclidic plane. Only that instead of complex numbers with $i^2=-1$ we have to use a hypercomplex system

$$z=z_1+e z_2 \tag{10}$$

where multiplication is defined by the convention

$$e^2=1. \tag{11}$$

We have already mentioned the theory of Conformal Relativity and we add § 8 on the basis of this theory. For some introductory words, the reader may consult the beginning of that section.

§ 1. Stereographic projection and I_∞

A clear picture of I_∞ can be obtained by stereographic projection as follows. Let Hy_n be the hyperboloid

$$X_1^2+X_2^2\ldots+X_{n-1}^2-X_n^2+X_{n+1}^2=1 \tag{1.1}$$

in the real $(n+1)$-dimensional space R_{n+1} with the metric

$$\bar{h}(X,X)=h(x,x)-X_{n+1}^2=-X_1^2\ldots-X_{n-1}^2+X_n^2-X_{n+1}^2 \tag{1.2}$$

and map the point $P=(x_1, x_2, \ldots x_n)$ of Lo_n on the intersection P' of Hy_n with a straight line $\boldsymbol{l}$ through $X=(x_1, \ldots x_n, 0)$ and the point

$$Q=(0, \ldots, 0, 1) \tag{1.3}$$

of Hy_n. The hyperboloid Hy_n is the "unit sphere" of R_{n+1}, and the construction corresponds to the stereographic projection of the complex plane on a unit sphere in three dimensions. The straight line $\boldsymbol{l}$ has in parameter form the equations

$$X_i=u x_i, \qquad i=1, 2, \ldots n, \qquad X_{n+1}=(1-u), \tag{1.4}$$

and it follows from (1.1) that $\boldsymbol{l}$ intersects Hy_n for the value $u=2/(1-s^2)$ corresponding to

$$P'\begin{cases} X_i=\dfrac{2x_i}{1-s^2} & (i=1, 2, \ldots n), \\ X_{n+1}=\dfrac{s^2+1}{s^2-1}. \end{cases} \tag{1.5}$$

The metric (1.2) induces on Hy_n a metric

$$dS^2=-dX_1^2\cdots+dX_n^2-dX_{n+1}^2\,.$$

which is actually one of the forms of de Sitter's line-element (Schrödinger [4]). That (1.5) defines a conformal mapping of Lo_n on Hy_n can be proved in different ways. If ds is an infinitesimal line-element in Lo_n, which by (1.5) is mapped on the element dS of Hy_n, straightforward elementary calculations, not reproduced here, show that

$$dS = \frac{2}{(1-s^2)}\, ds. \tag{1.6}$$

The fact that this formula is independent of the orientation of ds shows that the mapping is conformal. It can also be noted that (1.5) has the character of an inversion in R_{n+1}. The distance S_1 between Q and the point p measured in the metric (1.1) is given by $S_1^2 = s^2 - 1$. But the distances along pQ are proportional to the u-value, which means that the distance $PQ = S_2$ is given by $S_2^2 = 4/(s^2-1)$. Thus we get the relation $S_1^2 S_2^2 = 4$, characteristic for an inversion with centre Q. An inversion in R_{n+1} is conformal in R_{n+1} and then of course the part of this inversion which maps Lo_n on Hy_n must be conformal, as the metric in both Lo_n and Hy_n is induced by the metric in R_{n+1}.

The mapping $Hy_n \to Lo_n$ transports Q to a point at infinity in Lo_n, but not only Q. All points on Hy_n satisfying the condition

$$X_{n+1} = 1 \tag{1.7}$$

give a line of projection $\boldsymbol{l}$ parallel to the hyperplane $X_{n+1} = 0$, that is, intersects Lo_n at infinity. But (1.7) is the tangent plane of Hy_n in Q, and we can identify Q with the ideal element P_∞ and the intersection of Hy_n and (1.7) with the cone of P_∞.

In order to justify this, we have to show that an arbitrary finite point P of Lo_n and its cone is mapped on a point P' of Hy_n and the intersection of Hy_n with the tangent plane in P'. As a matter of fact, if $x = x_i$ and $x' = x_i'$ are two points in Lo_n, on a null-line, it is easy to prove that the corresponding points $X = X_i$ and $X' = X_i'$ of Hy_n are on a null-line in R_{n+1}. For we then have $2 x_i x_i' = s^2 + s'^2$ and

$$\sum_i \left(\frac{2 x_i}{1-s^2} - \frac{2 x_i'}{1-s'^2}\right)^2 - \left(\frac{s^2+1}{s^2-1} - \frac{s'^2+1}{s'^2-1}\right)^2 = 0$$

is satisfied identically. At the summation signs are adjusted to (1). Null-lines on Lo_n are mapped on null-lines of Hy_n, but on the other hand each null-line of Hy_n belongs to the tangent plane of each of its points, as from

$$(X_1 - X_1')^2 \ldots - (X_n - X_n')^2 + (X_{n+1} - X'_{n+1})^2 = 0$$

follows $$X_1 (X_1 - X_1') \ldots - X_n (X_n - X_n') + X_{n+1} (X_{n+1} - X'_{n+1}) = 0,$$

which is the condition for $\mathbf{X} - \mathbf{X}'$ to be a tangent vector of Hy_n in X.

§ 2. Hexaspherical coordinates and the space R_{n+2}

We have found that it is justified to speak of a "cone at infinity" defined in R_{n+1} by

$$-X_1^2-X_2^2\ldots-X_{n-1}^2+X_n^2=0, \qquad X_{n+1}=1. \tag{2.1}$$

However, even if Hy_n is well adapted for a study of the neighbourhood of the point P_∞, it is not a complete map of Lo_n', as is seen from the second equation (1.5). All points of Lo_n' for which $s^2=1$ are found at infinity on Hy_n. That is the reason why the compactification of Lo_n and Hy_n can only be completed with the aid of the space R_{n+2} defined in the introduction.

We obtain at once the hexaspherical coordinates (4) and the metric (3) if we introduce homogeneous coordinates in R_{n+1} through

$$X_i=\frac{y_i}{y_{n+2}} \qquad (i=1, 2, \ldots, n+1). \tag{2.2}$$

Now, all points for which $s^2=1$, are again included, as they correspond to $y_{n+2}=0$. All points of Lo_n are characterized by $y_{n+1}\neq y_{n+2}$, and in the introduction we have already described how the compactifical Lo_n' is obtained by inclusion of idealized elements corresponding to rays in R_{n+2} with $y_{n+1}=y_{n+2}$.

Inversions in Lo_n' as reflections in R_{n+2}

We can write each Lorentz-sphere in the form (6) or in K_n in the form

$$\sum_1^{n+2} a_i y_i=0, \qquad a_{n+1}+a_{n+2}\neq 0. \tag{2.3}$$

Here we can regard a_i as covariant components of a vector $\mathbf{a}$. For applications in Conformal Relativity compare Ingraham [5]. The contravariant components a^i are obtained by changing the sign of all a_i except a_n and a_{n+2}. If we norm the vector $\mathbf{a}$ so that

$$a_{n+1}+a_{n+2}=1 \tag{2.4}$$

we find immediately from (6) that the radius r of the sphere is

$$r^2=H(a, a), \tag{2.5}$$

whereas $(a^1, a^2, \ldots, a^n)$ is the centre of the sphere. We shall prove that *a reflection in the plane* (2.3) *corresponds to an inversion in the sphere* (6).

If $\mathbf{e}$ is the unit vector $\mathbf{e}=\mathbf{a}/r$, the reflection is defined by

$$\mathbf{y}'=\mathbf{y}-2(e\,y)\,\mathbf{e} \tag{2.6}$$

with $$(e\,y) = \frac{1}{r} \sum a_n y_n . \tag{2.7}$$

Obviously $H(y', y') = 0$ if $H(y, y) = 0$.

On the ray specified by y_i, we take in (2.6) y as the vector (4) characterized by the condition $y_{n+2} - y_{n+1} = 1$. To obtain from y' the components x_i' of the corresponding point in Lo_n, we have to divide y' with the number

$$\lambda = y'_{n+2} - y'_{n+1} = 1 - \frac{2}{r}(e\,y). \tag{2.8}$$

But $$h(x-a, x-a) = h(x, x) + h(a, a) - 2\sum_1^n a_i x_i$$

$$= s^2 + r^2 + a_{n+1}^2 - a_{n+2}^2 - 2\sum_1^n a_i y_i ;$$

it follows the indentity $$h(x-a, x-a) = r^2,$$

$$x_i' = \frac{y_i'}{\lambda} = \frac{x_i - a_i - \frac{2}{r}(e\,y)}{\lambda} .$$

Hence $$(x' - a) = \frac{(x-a)}{h(x-a, x-a)} r^2 \qquad \text{Q.E.D.}$$

If we allow $a_{n+1} + a_{n+2} = 0$ instead of (2.5) we obtain all hyperplanes in Lo_n' instead of spheres. All hyperplanes, as well as all spheres, are topologically closed in Lo_n', and all points of such a closed figure are equivalent conformally. Each sphere, space-like or time-like, has as tangent cone the cone of the centre of the sphere and intersects I_∞ in the same $(n-2)$-dimensional manifold as this cone. Each space-like hyperplane on the contrary intersects I_∞ in only one point, as is made clear in the next section, whereas each time-like hypersphere intersects I_∞ in an $(n-2)$-dimensional manifold.

§ 3. Straight lines, null-lines and null-cones in Lo_n'

Each straight line l is a *closed* curve in Lo_n' and intersects I_∞ at one point which is not difficult to calculate. Suppose that l goes through a finite point with coordinates x_i and has a direction given by a vector (e^i). In parametric form it has then in Lo_n the coordinates $x_i + u\,e^i$ and in R_{n+2} coordinates

$$x_1 + u\,e^1, \ldots, x_n + u\,e^n, -\tfrac{1}{2}(1+s^2), \tfrac{1}{2}(1-s^2) \tag{3.1}$$

with $$s^2 = h(x - u\,e, x - u\,e) = h(x, x) + u^2 h(e, e) - 2\,u\,(e\,x). \tag{3.2}$$

To find the limit of (3.1) as $u \to \infty$ we see from (3.2) that we must distinguish the two cases $h(e,e) \neq 0$ and $h(e,e) = 0$. In the first case $\boldsymbol{l}$ is not a null-line, we have to divide in (3.1) with u^2 and find after $u \to \infty$ that y_i are proportional to the vector

$$(0, 0, \ldots, 0, 1, 1) \tag{3.3}$$

which represents the point P_∞. *Hence all straight lines in* Lo_n' *which are not null-lines, are closed curves, "circles", passing through the vertex* P_∞ *of the cone at infinity.*

But if $h(e,e) = 0$ we must according to (3.2) divide by u and then find that (3.1) as $u \to \infty$ approaches the vector

$$(e^1, \ldots, e^n, (e\,x), (e\,x)) \tag{3.4}$$

with

$$(e\,x) = \sum_1^n e_i\,x_i \qquad (e_i \text{ covariant comp.}). \tag{3.5}$$

This is a well-defined point of I_∞. The most interesting feature is that the point (3.4) is *not* uniquely determined by the direction $\mathfrak{e}$ of the null-line. The appearance of $(e\,x)$ means that parallel null-lines can determine different points of I_∞. Two parallel null-lines obviously determine the same point of I_∞ if and only if they belong to *the same singular hyperplane* given by

$$\sum_1^n e_i\,x_i = \text{const.} \tag{3.6}$$

The situation is really very different from what could be expected at first sight. Compare with projective space, where all parallel lines determine the same point at infinity. In a Lorentz-space all ordinary lines, parallel or not, determine one point P_∞, whereas all null-lines in the same singular plane determine one point of I_∞.

But we understand at once the consistency of this, if we ask for the cone of a point P of I_∞. If $P \neq P_\infty$ this cone, according to the above, is actually identical with the *singular* hyperplane (3.6). This is also natural from the point of view that singular hyperplanes Pl are the limits of ordinary cones, as is seen by considering the cone of a point P of Pl when $P \to \infty$ along a null-line. Compare with the case $n = 3$, where conditions are easily visualised.

To summarize: *The compactification of* Lo_n *implies at the same time a compactification of the light cones through adjunction of all singular hyperplanes and one* ∞*-cone.*

However, two cones C_1 and C_2 with vertex P_1 and P_2 have always a common null-ray if P_1 lies on C_2 and P_2 on C_1. The situation is the same for the ∞-cone I_∞ and a singular hyperplane with vertex P. The common null-geodesic is, of course, the one which connects P and P_∞. This is the reason why all

straight lines on a singular hyperplane, which are not parallel with the isotropic normal $\mathfrak{e}$, can intersect I_∞ in P_∞.

If Pl is a space-like hyperplane, then all straight lines in Pl intersect I_∞ in P_∞ and it follows that Pl as a whole intersects I_∞ at the single point P_∞, just as Pl intersects each light cone at a point P of Pl only at the point P itself.

If Pl is a time-like hyperplane it contains null-geodesics and thus intersects I_∞ at points $\neq P_\infty$. Actually, it intersects I_∞ in an $(n-2)$-dimensional manifold of null-geodesics, just as for every other cone in a point of Pl.

§ 4. The conformal transformation as composed of similarity-transformations and inversions

In the case of Euclidean geometry, each conformal transformation can be regarded as a combination of an inversion, transforming the desired point to infinity, and a translation, a rotation, an enlargement—the last three summarized as a "similarity-transformation".

In the case of Lo_n' the situation is somewhat different as there are many ∞-points. If we wish to transfer a finite point P to the vertex of the ∞-cone, we need only one inversion, as in the Euclidean case. But if P is on the ∞-cone, *two* inversions are necassary, the first one is arbitrary, the second one has the now finite point P as centre.

If on the other hand we have transformed two points P_1 and P_2 (which can be chosen arbitrarily in Lo_n' except that P_1P_2 must not be a null-ray) to P_∞ and the origin, the remaining transformation with P_∞ and the origin as fix-points must be an enlargement and a Lorentz transformation. The existence of I_∞ makes it necessary to prove this statement.

Consider all pseudo-Euclidean transformations in R_{n+2} with two invariant null-rays $(0, \dots 0, \pm 1, 1)$. It follows immediately that the transformation matrix consists of two boxes, namely, a Lorentz matrix for $(y_1 y_2 \dots y_n)$ and a Lorentz matrix $\begin{pmatrix} a, b \\ b, a \end{pmatrix}$ for y_{n+1}, y_{n+2}. The first box corresponds to a Lorentz-matrix in Lo_n. On the other hand, a pseudo-rotation of the y_{n+1}, y_{n+2}-axis leaves all the proportions $y_1 : y_2 : \dots : y_n$ invariant, which according to (4) means that the proportions $x_1 : x_2 : \dots : x_n$ are also invariant, or in other words $x_i' = k \cdot x_i$. It is easy to prove that the connection between a, b and k is given by

$$a = \tfrac{1}{2}\left(k + \frac{1}{k}\right), \qquad b = \tfrac{1}{2}\left(k - \frac{1}{k}\right). \tag{4.1}$$

The fact that some conformal transformations of a Lorentz space Lo_n' require *two* inversions, does not mean that the number of parameters is larger

than in the Euclidean case with the same number of dimensions. In both cases, the general conformal transformation is reduced to a linear pseudo-Euclidean transformation in the same number of dimensions and can, if e.g. $n=4$, be composed of 15 simple rotations in R_6.

§ 5. The topological structure of Lo'_n

This problem can be dealt with in different ways, and we choose here a method which at the same time gives some further insight into the analytical tools. To begin with, it must be noted that $\mathcal{G}_{n+2}$ as well as $\mathcal{G}_{n+2}/\{E, -E\}$ like the full Lorentz group consists of several disconnected parts. Let us call the continuous part containing the identity the "principal part".

We then note that among all inversions described in § 2, the inversion in the unit circle is particularly simple. It is in Lo'_n defined by

$$\left.\begin{aligned} x'_i &= \frac{x_i}{s^2} \qquad (i=1, 2, \ldots n-1), \\ x'_n &= -\frac{x_n}{s^2} \end{aligned}\right\} \tag{5.1}$$

and corresponds in the part K_n of R_{n+2} to the reflection

$$y'_i = y_i \quad (i=1, 2, \ldots n-1, n+1), \quad y'_n = -y_n, \quad y'_{n+2} = -y_{n+2}. \tag{5.2}$$

Hence, the inversion (5.1) belongs to the principal part of $\mathcal{G}_{n+2}$.

In the space R_{n+1} analyzed in § 1 (5.1) and (5.2) corresponds to

$$X'_i = -X_i \quad (i=1, 2, \ldots n-1, n+1), \quad X'_n = X_n. \tag{5.3}$$

It follows that for n *even*, (5.1) can be obtained as a rotation (actually Euclidean) in the space R_{n+1}. For n odd this is not possible.

To get a clear picture of orientability, we investigate in more detail the conformal transformation corresponding to

$$\left.\begin{aligned} y'_n &= \cos\varphi\, y_n - \sin\varphi\, y_{n+2}, \\ y'_{n+2} &= \sin\varphi\, y_n + \cos\varphi\, y_{n+2}, \end{aligned}\right\} \tag{5.4}$$

which connects the identity with (5.2) as is seen by putting $\varphi=\pi$. By these transformations the proportions $y_1 : y_2 \ldots y_{n\ 1} : y_{n+1}$ are unaffected for all φ and this means that the intersection between $H y'_n$ and an arbitrary 2-plane through the X_n-axis is mapped on itself (Fig. 1). How this mapping is performed follows from (5.4) which gives

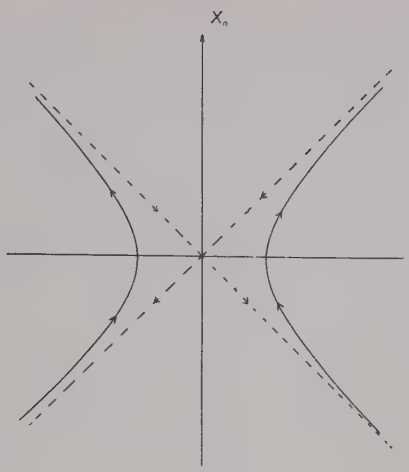

Fig. 1.

$$X_n' = \frac{\cos\varphi\, X_n - \sin\varphi}{\sin\varphi\, X_n + \cos\varphi}. \tag{5.5}$$

All the time-like "circles" under consideration are thus "rotated" the same angle, but rotation has of course not the same geometrical meaning as in Euclidean geometry, as is indicated by arrows in Fig. 1.

We now introduce a small frame of reference K in, say, the origin O, and investigate what happens to the orientation of K, when O and with it K is moved by a continuous one-parametric sequence of transformations of $\mathcal{G}_{n+2}$. More precisely, we consider a continuous curve in the parameter-space of $\mathcal{G}_{n+2}$. When $g \subset \mathcal{G}_{n+2}$ describes this curve, O (and K) describes a corresponding curve $\mathfrak{L}$ on Hy_n'. As long as $\mathfrak{L}$ do not pass through an infinity-point of Hy_n, the orientation of K cannot change. This follows from the simple topological structure of Hy_n, which has the character of a cylinder.

But if the curve $\mathfrak{L}$ passes through a point at infinity of Hy_n', that is, through a point with $y_{n+2} = 0$ in K_n, the orientation can be changed. Explicitly, we can then take (5.4) with $\varphi = 0 \ldots \pi$ as part of the curve $\mathfrak{L}$. Then, if n is even, we can make a rotation in R_{n+1} changing signs for all X_i, $i = 1, 2, \ldots n-1, n+1$, and we recover according to (5.3) the identical transformation, that is, the orientation is not changed. But, if n is odd, we can instead make a rotation which changes sign of X_i only for $i = 2, \ldots n-1, n+1$ and we thus obtain the transformation $X_1' = -X_1$, $X_i' = X_i$ $(i = 2, 3, \ldots n+1)$. This corresponds in Lo_n' to the transformation.

$$x_1' = -x_1, \qquad x_i' = x_i \qquad (i = 2, \ldots n). \tag{5.6}$$

Hence the orientation is changed, although we have followed a closed continuous curve in the group $\mathcal{G}_{n+2}$.

We have already mentioned that topologically Hy is a cylinder, being a product of a hypersphere with dimension $(n-1)$ and a line. If we are only interested in the topology, we can instead of X_n use a parameter t varying in the interval $0<t<1$, and denote the corresponding spheres with S_t. The compactification of Hy_n means that we add a sphere S_0 and a sphere S_1, which are then identified. More precisely, each point P of S_0 is identified with the diametrically opposite point P' of S_1. It is clear from this construction that orientability is the same as in $(n-1)$-dimensional projective space.

The full Lorentz group in n dimensions consists of four disconnected parts, distinguished by orientation of the space-like axes and by the direction of the positive time-axis. Now the full Lorentz group is of course a subgroup of the full conformal group C_n. But as we have found that the space-axes are non-orientable if n odd, we conclude at once: *The full conformal group C_n consists of four disconnected parts if n even, but of two disconnected parts if n odd.*

Note on spin representations of the conformal group

In possible physical applications linear representations of the conformal group must play a leading part (compare e.g. Murais [6]). If we restrict ourselves to the principal part, the two-valuedness of spin representation of the Lorentz group is a consequence of the existence of closed curves in the parameter space of the group, which cannot be continuously contracted to a point, e.g. the rotation of ordinary 3-dimensional Euclidean space the angle 2π with x_3 as axis.

However, as far as I know, it has not been observed that the situation is slightly different for the principal part of C_n. *If n even the spin representation is actually four-valued.* The topological structure of Lo'_n or, what is the same, Hy'_n allows *two* different kinds of closed curves on Hy'_n, which cannot be deformed to a point, just as on an ordinary torus. Correspondingly, we have two different kinds of closed curves in the parameter space of C_n. One type is given by (5.4) and the other type by a rotation the angle 2π in e.g. the 2-plane spanned by the $y_1 y_{n+1}$-axes.

Algebraically, we obtain of course the spin representations of C_n from the spin representations of the rotation group G_{n+2}, (compare introduction). Then the element $-E$ of G_{n+2}, or $y'_i = -y_i$, corresponds to *two* matrixes in a spin representation, the same being the case for E. But E and $-E$ both represent the identity-transformation in C_n, which means that the spin representation is four-valued in C_n.

§ 6. The two-dimensional case

As an application of the foregoing analysis, we shall in this section regard the compactification of a Lorentz plane Lo_2 and the corresponding group of

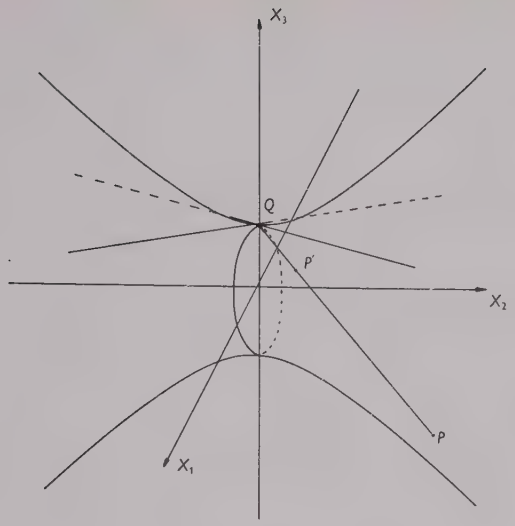

Fig. 2.

Lorentz-conformal transformations C_2. Instead of using indices, we take (x, t) as space- and time-coordinates in Lo_2. The space R_{n+1} is now three-dimensional with X_1, X_2, X_3 as coordinate-axes. Lo_2 can, if we wish, be identified with the plane $X_3 = 0$ and the hyperboloid Hy_2 has the equation

$$X_1^2 - X_2^2 + X_3^2 = 1. \tag{6.1}$$

The centre of the stereographic projection described in § 1 is the point $Q = (0, 0, 1)$ (compare Fig. 2). If $P = (x, t)$ is a point in Lo_2 the corresponding point P' on Hy_2 is obtained from (1.5). There are on Hy_2, as is well known from elementary analytical geometry, two families of rectilinear generators. Through each point P' goes two generators, one in each family, and they are projected on the two null-rays through P in the Lorentz plane Lo_2.

The plane $X_3 = 1$ intersects Hy_2 along the two generators in Q, and Q, as well as these two generators, corresponds to idealized points of Lo_2'. Conditions in the neighbourhood of P_∞ of Lo_2' are correctly described by conditions in the neighbourhood of Q on Hy_2. Consider for instance an arbitrary space-like straight line $\boldsymbol{l}$ in Lo_2. $\boldsymbol{l}$ is projected on Hy_2 by a plane through Q, intersecting Hy_2 along an ellipse (consider e.g. in Fig. 2 the image on Hy_2 of the x-axis). In the same way, an arbitrary time-like line in Lo_2 is projected on a hyperbola through Q (consider e.g. the image of the time-axis). Consider, on the other hand, an arbitrary generator $\boldsymbol{l}$ on Hy_2 of one family. $\boldsymbol{l}$ intersects the generator $\boldsymbol{l}'$ of the other family through Q in one point P'. The corresponding plane of projection is identical with the plane determined by $\boldsymbol{l}'$ and $\boldsymbol{l}$. From this construction, we understand why all non-isotropic straight lines in Lo_2

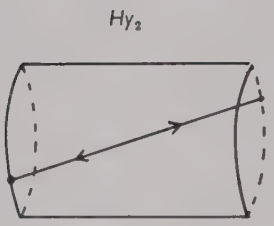

Fig. 3.

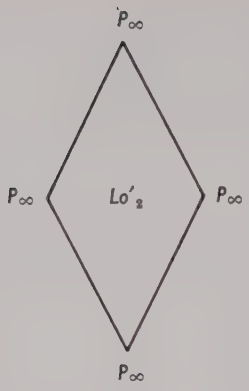

Fig. 4.

intersect the "cone at infinity" in the vertex P_∞ (or Q), whereas each null-line intersects the ∞-cone at another well-determined point, the point P' of the construction just mentioned.

Space-like lines of Lo_2' correspond to closed corves on Hy_2 and the same is the case for all but one space-like circles of Lo_2', which are obtained as the intersection of Hy_2 with all planes forming an angle $> \frac{1}{2}\pi$ with the X_2-axis. The only circle left out is of course $s^2 = t^2 - x^2 = 1$. Time-like lines of Lo_2' are not closed curves on Hy_2, but are at once transformed to closed curves, if we compactify Hy_2 by adding to it one space-like circle at infinity. Except for one point, this circle $s^2 = 1$ is found in Lo_2.

Topologically, we can describe Hy_2 as a 2-dimensional cylinder of finite length and constant radius (Fig. 3). The two circles as the end of the cylinder are both identical with $s^2 = 1$. They have to be identified as indicated in the figure and the cylinder is transformed to a "twisted torus". As a matter of fact it follows from § 5 that the compactifical Lo_n' always can be described as a kind of generalized twisted torus *if n is even.*

The twisting sometimes makes it difficult to imagine how a certain figure of Lo_2' is reproduced on Hy_2'. It may therefore be worth noting that it is also permitted to regard Lo_2' as an ordinary torus, or a parallelogram with opposite points of the sides identified. In the last picture we can, if we wish, identify all null-rays with the two families of straight lines parallel to the sides (Fig. 4). The connection between Lo_2' as mapped on Hy_2', or as an ordinary torus, is perhaps made most clear by considering the universal covering surface (Fig. 5). In the figure inclined lines can be regarded as idealized null-rays of Lo_2'. If we cut Hy_2' along $s^2 = 1$ and one branch of the hyperbola $X_3^2 - X_2^2 = 1$, we obtain a rectangle and in Fig. 5 such rectangles are joined together just as the twisting of the torus requires. We understand from the figure how a non-twisted torus can be formed from a rectangle with idealized null-rays as sides. But it must be remarked that no corresponding construction is possible for $n > 2$.

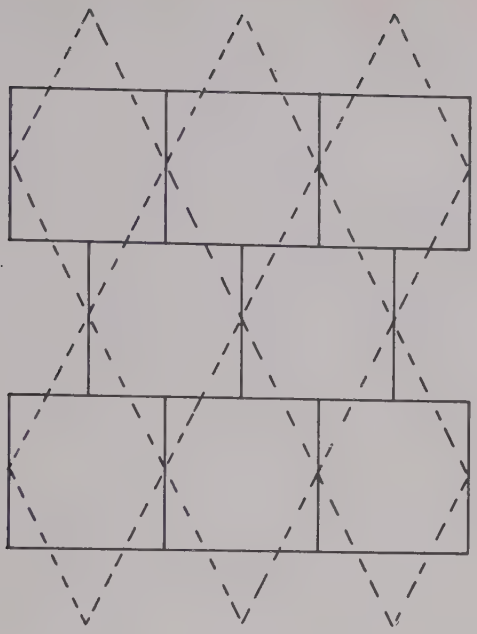

Fig. 5.

§ 7. Spherical transformation in Lo_2'

Following the general scheme we obtain all spherical transformations of Lo_2' from the pseudo-orthogonal transformations of R_4 with the metric

$$h(y, y) = -y_1^2 + y_2^2 - y_3^2 + y_4^2. \tag{7.1}$$

We note that the classification of these transformations constitute a rather interesting problem, which has been solved by H. Jacobinski [7] for a much more general metric.

Referring to the introduction we now consider hypercomplex or, as we shall call them, "hyperbolic" numbers $z = z_1 + e z_2$ with multiplication defined by the convention $e^2 = 1$. They can be regarded as a Clifford algebra of the simplest type, and, contrary to other Clifford algebras, this one is commutative, as there is only one basis-element, which of course commutes with 1. The "norm" of z is defined by

$$|z| = \sqrt{|z_2^2 - z_1^2|}. \tag{7.2}$$

We investigate the geometrical meaning of the transformation

$$w = a z \tag{7.3}$$

if $a = a_1 + e a_2$ is a hyperbolic number with unit-norm satisfying $a_1 > |a_2|$. We find at once

$$\left.\begin{aligned} w_1 &= a_1 z_1 + a_2 z_2, \\ w_2 &= a_2 z_1 + a_1 z_2, \end{aligned}\right\} \tag{7.4}$$

which is a Lorentz transformation, if we identify z_2 with the time- and z_1 with the space-axis. If $a = a_1$ is a "real" positive number, then (7.3) is an enlargement and as each number a can be written as the product of a real number and a number with unit-norm, we conclude that (7.3) is a similarity transformation in the Lorentz-plane with the origin as fix-point.

We proceed and regard

$$w = -\frac{1}{z} \tag{7.5}$$

which is defined only if $|z| \neq 0$. We then find

$$w = \frac{+z_1 - e z_2}{z_2^2 - z_1^2}$$

or

$$w_1 = \frac{z_1}{s^2}, \qquad w_2 = -\frac{z_2}{s^2}, \tag{7.6}$$

that is, an inversion in the unit-circle. But each conformal transformation, with the added property of transforming circles into circles, can be composed of inversions and similarity transformations. Combining (7.3), (7.5) and a translation $w = a + z$ we conclude that the group C_2 consists of all transformations, which can be written in the form

$$\left.\begin{aligned} w &= \frac{a z + b}{c z + d}, \\ D &= a d - b c \qquad (D \neq 0), \end{aligned}\right\} \tag{7.7}$$

just as in the domain of complex numbers the Möbius group gives us all Euclid-conformal transformations of a plane.

Bearing in mind the properties of $L o_2'$, we can however expect that a given transformation (7.7) cannot always be decomposed exactly in the same way as in the Euclidean case, as we know that sometimes *two* inversions are necessary to transform the desired point to P_∞. First we have to observe the following point. Eq. (7.4) can be written

$$w = \frac{\sqrt{a}\, z}{1/\sqrt{a}},$$

which means that $D = a d - b c = 1$. For (7.5) as well as $w = a + z$ we have also

$D=1$. But at the composition of two transformations of the type (7.7), the constant $D=D_1 D_2$ and we can therefore restrict (7.7) by the condition

$$ad-bc=1. \tag{7.8}$$

Then if $|c|\neq 0$ we can write (7.7) in the form

$$w=\frac{a}{c}-\frac{1}{c}\,\frac{1}{cz+d}, \tag{7.9}$$

and this means that (7.7) can be decomposed with the aid of two translations, *one* inversion and one transformation of type (7.3).

The remaining case $|c|=0$ is precisely the one which requires *two* inversions, as is at once found by combining a transformation $w=b'/(z+d')$ with a translation c followed by another inversion. We conclude that (7.7) with the condition (7.8) gives precisely all transformations in the principal part of C_2.

It is perhaps worth noting that from (7.7) we also easily obtain the spin representations of C_2 replacing (7.7) by two linear equations with $\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$ as transformation matrix. We thus get a spin representation in terms of hyperbolic numbers, from which of course a corresponding representation in real numbers is easily deduced.

§ 8. On the basis of the conformal theory of relativity

In the paper of J. A. Schouten [1] a rather detailed exposition of the development of the theory of conformal relativity is given. Here we only note that as early as 1910 Cunningham [11] and Bateman found that the equations of Maxwell are invariant for the 15-parametrical group of conformal transformations of Lo_4. Later, Haantjes and Hill [8, 9] have found a connection between conformal transformations and physical observers with *constant acceleration*. The idea is thus to formulate laws of nature which are not only invariant in the ordinary relativistic sense, with regard to all observers passing a world-point P with constant velocity, but also to the much wider class of observers passing P with constant acceleration.

If we consider a conformal transformation of Lo_4' with the origin O as fixpoint, then the time-axis x_4 is transformed to another time-like circle through O, analytically this "circle" is a hyperbola if described in the original coordinates, and it is well known from the theory of relativity that the world-line followed by an observer with constant acceleration is actually a time-like hyperbola (Møller [10]).

Freeing ourselves from the historical development, we can ask what are the simplest physical assumptions which logically result in the idea that all obser-

vers with constant accceleration are equivalent. I shall in this section give a few views on this problem.

At the very basis of Einstein's analysis of the concept of simultaneity we find the picture of an observer A emitting light-signals, which after reflection at a point P are again received by A. The times t_1 of emitting and t_2 of receiving serve as measurables with the aid of which A calculates time and distance of P. Usually, this elementary experiment is only thought of as taking place in a rudimentary universe with one space dimension and so there is no problem with angles between light-rays.

The "angle postulate"

If however we conceive an observer A in four-dimensional space-time, trying to introduce coordinates in his surroundings, with the aid of light signals and measurements of proper time along his own world-line, then we have at least to require that if A emits light in one direction, then the reflected light-ray shall return *from the same direction.* Or if we make things a little more complicated by allowing a rotation of A's physical system, we must require that *two* light signals which at the same time are emitted in two directions forming the angle φ with each other, if they later return simultaneously, always return from two directions which still form the angle φ with each other. If this condition is fulfilled A can correct all calculated coordinates for his own rotation; if it is not valid, A will probably be at a loss to understand the geometry of the world in his surroundings.

I shall show below that this somewhat loosely-formulated postulate actually leads directly to the class of observers with constant acceleration and thus to the conformal group C_4.

One more speculative thought may be allowed. In the actual universe an observer can, by internal experiments in an enclosed laboratory, verify whether he moves with constant velocity (free fall) or not. If we ask *why* some world-lines can physically be regarded as accelerated and others not, then we must answer that this is a consequence of the influence of the universe as a whole. But what would happen in a universe built up by only a few material bodies, when there is no sky of "fixed stars" providing us with preferred systems of reference? Yet the actual universe can be regarded as a limiting case of such a "small" universe. It is then of interest that the above-mentioned standard experiments with light-rays can, in principle, of course be carried through, even in a "small" universe. But what can they reasonably tell about preferred motions and systems of coordinates? To find a conclusive answer to such a vague question must be difficult, but what I shall try to prove below with elementary means is that it cannot be possible to distinguish with light signals between observers with constant relative acceleration.

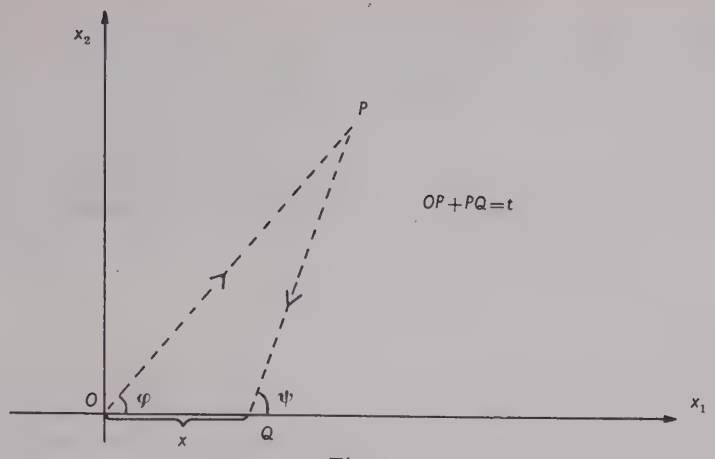

Fig. 6.

Let A be an observer at rest in the origin of a space-like plane, which according to his measurements is Euclidean with x_1, x_2, as rectilinear coordinate-axes. Let B be a second observer who at the time $x_4 = 0$ is also momentarily at rest at the origin, but with this initial condition possibly describes some path in the $x_1 x_2$-plane. Suppose for the sake of simplicity that B moves along the x_1-axis. At the origin, a light-ray $\boldsymbol{l}$ is emitted in the direction φ (Fig. 6). Since B at rest at the origin φ can be taken as the measure of the angle both in the system of A and B. The light-ray $\boldsymbol{l}$ is reflected at a point P, somewhere in the $x_1 x_2$-plane and then returned to B, which however at receipt of $\boldsymbol{l}$ has moved, say the distance x, measured in the system of A. Suppose that the returning light-ray forms the angle ψ with the positive x_1-axis *measured in the A-system.* But now B moves with a certain velocity, relative to A and therefore measures another angle. We ask—*what is the velocity of B if he instead of ψ measures the same angle φ which was the starting-direction of $\boldsymbol{l}$ in O?* As a consequence of the phenomenon known as abberation of light, the velocity of B (also its direction) is uniquely determined by this condition.

If we denote the projection of OP on the x_1-axis with m, we find

$$\left.\begin{aligned} \frac{m}{\cos\varphi} + \frac{m-x}{\cos\psi} &= t, \\ m \operatorname{tg}\varphi &= (m-x)\operatorname{tg}\psi, \end{aligned}\right\} \tag{8.1}$$

where $t = OP + PQ$ is the time which the signal measured in the A-system requires. If we introduce half angles $\frac{1}{2}\varphi$ and $\frac{1}{2}\psi$ it is only a question of some elementary calculations to show that from (8.1) follows

$$\frac{\operatorname{tg}\frac{1}{2}\varphi}{\operatorname{tg}\frac{1}{2}\psi} = \frac{t+x}{t-x}. \tag{8.2}$$

But, on the other hand, the formula for the abberation of light can be written in the form

$$\frac{\operatorname{tg} \frac{1}{2}\varphi}{\operatorname{tg} \frac{1}{2}\psi} = \sqrt{\frac{1+v}{1-v}}, \tag{8.3}$$

if $v = dx/dt$ is the velocity of B in the system of A. Combining (8.2) and (8.3) we find at once for the word-line of B the differential equation

$$\frac{dt}{dx} = \frac{1}{2}\left(\frac{t}{x} + \frac{x}{t}\right). \tag{8.4}$$

The solution of (8.4) is

$$t^2 = x^2 + px, \tag{8.5}$$

where p is the constant of integration. This means that we obtain *all* time-like hyperbolas in the x_1x_4-plane which at the origin have the x_4-axis as tangent. On the other hand we can, instead of A as a system of reference, take an arbitrary system of Lorentz-coordinates with the same origin and we can also choose an arbitrary space-like direction instead of the x_1-axis as direction of the movement of B, and so we obtain as word-lines of equivalent observers *all* hyperbolic motions passing through the origin. (It should be observed that the velocity given by (8.4) is the only possible one at the point Q if the angle postulate is to be satisfied.)

But we have not yet obtained all possibilities offered by the conformal group C_4, or more precisely the part of C_4 with the origin as fix-point. Suppose that the world-line of an observer A is given and identical with the x_4-axis. We then have the possibility of transforming Lo_4' conformally into itself in such a way that *the x_4-axis is mapped into itself with the origin as fix-point.* This means a change of the time-scale and the question is what can be said about this from a physical point of view.

In the ordinary theory of relativity, we are allowed to regard the method of measurement of proper time and the unit employed as fixed by rigorous physical conventions. Suppose that for some reason such conventions cannot at all be defined so that on the contrary the time-scale (not only the unit) is arbitrary to a large extent. What can then be achieved only by an appeal to the results of light signals? The coordinates which an observer A assigns to a worldpoint P with the aid of light signals fundamentally depend of course on his time-scale. We restrict our attention again to the case with one space-dimension and suppose that t_1 and t_2 is the time of starting and time of return for a light signal, reflected at a point P. The time- and space-coordinates x, t, which the observer A then associates with P, are

$$\left.\begin{aligned} x &= \tfrac{1}{2}(t_2 - t_1), \\ t &= \tfrac{1}{2}(t_2 + t_1). \end{aligned}\right\} \tag{8.6}$$

The time-scale postulate

We suppose that at least for one choice of time-scale all curves $x = \text{const}$ are world-lines of observers for which the "angle postulate" is satisfied, and we call all time-scales with this property "allowed".

If the t-values in (8.6) refer to one allowed time-scale, we can determine all transformations $t' = F(t)$ to another allowed time-scale. The equation $x = \text{const}$ is, if we employ the scale t' instead, replaced by the equation

$$\tfrac{1}{2}[F(t_2) - F(t_1)] = \text{const}$$

or

$$F(t+x) - F(t-x) = \text{const}. \tag{8.7}$$

The condition on F is that (8.7) shall be a time-like Lorentz circle. Then t, as function of x, must satisfy

$$\frac{dt}{dx} = \frac{1}{2}\left[\frac{t-h}{x-k} + \frac{x-k}{t-h}\right]. \tag{8.8}$$

Derivation of (8.7) with regard to x gives

$$F'(t+x)\left(\frac{dt}{dx}+1\right) - F'(t-x)\left(\frac{dt}{dx}-1\right) = 0 \tag{8.9}$$

or

$$F'(t+x)(t-h+x-k)^2 = F'(t-x)(t-h-x+k)^2. \tag{8.10}$$

Here we insert $t = x = \frac{1}{2}u$ and obtain for F a differential equation of form

$$\frac{dF}{du} = \frac{1}{(cu+d)^2} \tag{8.11}$$

from which follows

$$t' = F(t) = \frac{at+b}{ct+d}. \tag{8.12}$$

Referring back to the general theory of conformal transformations in Lo_4', we can conclude that (8.12) is precisely the subgroup of transformations mapping the x_4-axis into itself. For the point of Lo_4' mapped on P_∞ is then a point on the x_4-axis, which means that only one inversion (or none) is required and has the centre on the x_4-axis. A translation, if it is necessary, is also a translation in the direction of the x_4-axis. All transformations of the time-scale obtained in this way are of the type (8.12).

We have thus been able to show that the "angle postulate" and the "time-scale postulate" given above, which are both quite natural from a physical point of view, are sufficient to characterize the subgroup of C_4 with the origin as fix-point as the group of coordinate-transformations induced by all equivalent observers passing the origin. Compare with ordinary relativity, where the corresponding group is the Lorentz group.

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